# OFDM Systems — Why Cyclic Prefix?

The details of OFDM transmitter and receiver [1] structure are succinctly presented in the block diagram below. We note that the OFDM systems basically involve transmission of a cyclic prefixed signal over a fading multipath channel. The prime goal of this tutorial is to explain the significance of *Cyclic Prefix*. Why is it necessary and what implications does it have?

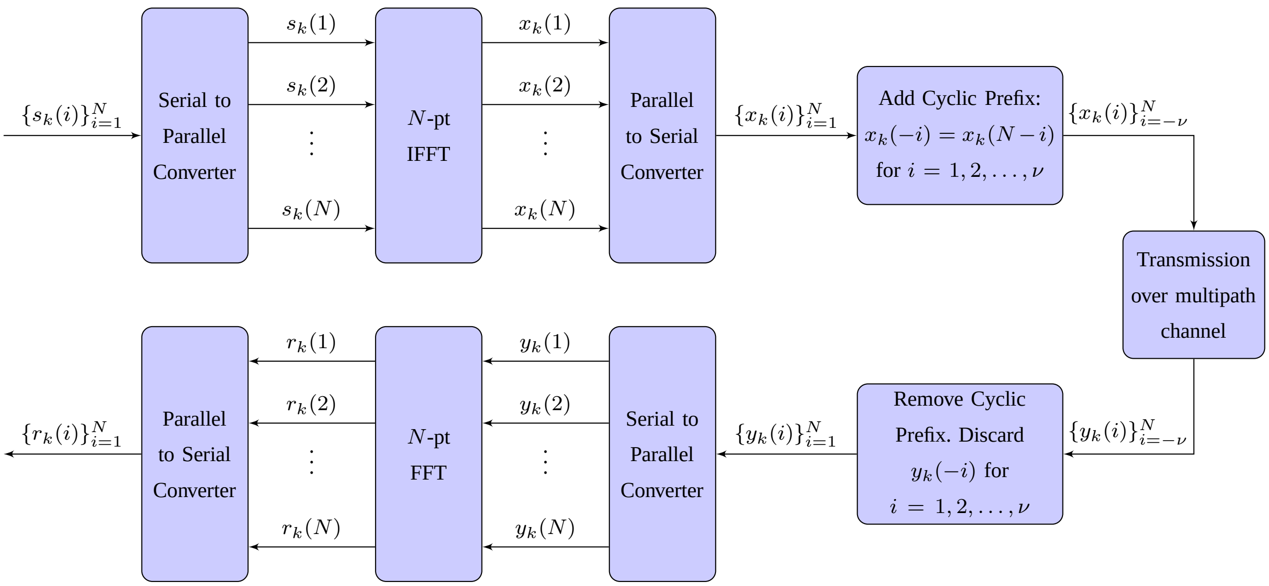


Fig. 1. Typical OFDM Transmitter - Receiver Structure.

The input symbols $\{s_k(i)\}_{i=1}^N$ denotes the transmit symbols for the $k-$th OFDM block. These symbols may come for instance from a M-QAM constellation. $N$ denotes the number of OFDM sub-carriers (the number of constellation symbols to be transmitted in one OFDM block). After serial to parallel conversion of the input symbol stream, a N-pt IFFT is taken to get[1] $\{x_k(i)\}_{i=1}^N$. After back parallel to serial conversion, a cyclic redundancy of length $\nu$ (the number of CP samples) is added as a prefix in such a way that $x_k(-i) = x_k(N-i)$ for $i = 1, 2, \ldots, \nu$.

Authored by Muhammad Danish Nisar who is with the Associate Institute for Signal Processing (MSV), Technische Universität München (TUM), 80290, Munich, Germany. (e-mail: mdanishnisar@ieee.org)

[1]Because of the traditional meaning of IFFT, the symbols $\{s_k(i)\}_{i=1}^N$ are often called *frequency domain symbols*, while $\{x_k(i)\}_{i=1}^N$ are accordingly called *time domain symbols*.

The signal is then transmitted on a multipath channel with the *Channel Impulse Response (CIR)* of the multipath channel of length $L$ denoted here by the vector

$$\boldsymbol{h} = \begin{bmatrix} h_0 & h_1 & \dots & h_{L-1} \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^L \tag{1}$$

The typical convolution equation for the $k_{\text{th}}$ channel output symbol

$$\boldsymbol{y}_k = \begin{bmatrix} y_k(0) & y_k(1) & \dots & y_k(N-1) \end{bmatrix}^{\mathrm{T}} \in \mathbb{C}^N \tag{2}$$

can be expressed in matrix notation in terms of transmitted samples $x_k(i)$ and noise vector $\tilde{\eta}_k \in \mathbb{C}^N$ as under,

$$\begin{bmatrix} y_k(0) \\ y_k(1) \\ \vdots \\ \vdots \\ \vdots \\ y_k(N-1) \end{bmatrix} = \begin{bmatrix} h_{L-1} & \dots & h_\nu & \dots & h_0 & 0 & \dots & \dots & \dots & 0 \\ 0 & \ddots & & \ddots & & \ddots & \ddots & & & \vdots \\ \vdots & & \ddots & & \ddots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & & & \ddots & & \ddots & & \ddots & 0 \\ 0 & \dots & \dots & \dots & \dots & h_{L-1} & \dots & h_\nu & \dots & h_0 \end{bmatrix} \begin{bmatrix} x_{k-1}(N-E) \\ \vdots \\ x_{k-1}(N-1) \\ x_k(-\nu) \\ \vdots \\ x_k(0) \\ \vdots \\ \vdots \\ x_k(N-1) \end{bmatrix} + \tilde{\eta}_k \tag{3}$$

where $E = L - \nu - 1$ is the channel length exceeding the duration of Cyclic Prefix $\nu$. The entities marked in red color appear only if CIR length $L$ exceeds the duration of CP, i.e. $E > 0$ and thereby contribute to what is called as *Inter-Symbol Interference (ISI)*.

For the simplicity of presentation, in the sequel we consider the CP length to be greater than CIR length and refer the interested reader to [2] for the discussion of insufficient CP scenario. The incorporation of CP property ($x_k(-i) = x_k(N-i)$ for $i = 1, 2, \dots, \nu$) for the case of CIR being shorter than the duration of CP leads to the following equation,

$$\begin{bmatrix} y_k(0) \\ y_k(1) \\ \vdots \\ \vdots \\ \vdots \\ y_k(N-1) \end{bmatrix} = \begin{bmatrix} h_0 & 0 & \dots & 0 & h_{L-1} & \dots & h_1 \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & & \ddots & h_{L-1} \\ h_{L-1} & & & \ddots & \ddots & & 0 \\ 0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \dots & 0 & h_{L-1} & \dots & \dots & h_0 \end{bmatrix} \begin{bmatrix} x_k(0) \\ x_k(1) \\ \vdots \\ \vdots \\ \vdots \\ x_k(N-1) \end{bmatrix} + \tilde{\eta}_k \tag{4}$$

Interesting to note here is the fact that the effective $N \times N$ channel matrix now gets circulant i.e. its rows are circularly shifted versions of each other. This results in major simplifications, described below, once the receiver, as shown in Figure 1, takes the FFT after CP removal. Note however, that this circulant nature of the effective channel matrix is void if the channel is time variant, because in that case the CIR coefficients appearing in a row (corresponding to a sample of the OFDM symbol) are potentially different than the CIR coefficients appearing in some other row.

Thus for the case of sufficient Cyclic Prefix (transition from equation 3 to 4) and time-invariant channel (channel matrix in 4 becoming truly circulant), the under consideration system can be described by the following relationship in the sub-carrier (frequency) domain,

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{H}_{\text{CIRC}}\boldsymbol{F}^{\text{H}}\boldsymbol{s}_k + \eta_k \tag{5}$$

where $\boldsymbol{F} \in \mathbb{C}^{N\times N}$ is the normalized Fourier matrix (unitary in nature i.e. $\boldsymbol{F}\boldsymbol{F}^{\text{H}} = \boldsymbol{I}_N$). The vectors $\boldsymbol{s}_k, \boldsymbol{r}_k, \eta_k \in \mathbb{C}^N$ are frequency domain versions of $\boldsymbol{x}_k, \boldsymbol{y}_k, \tilde{\eta}_k \in \mathbb{C}^N$ and are obtained by linear transformations via the Fourier matrix, as evident from the diagram. Now because the *Eigen Value Decomposition (EVD)* of a circulant matrix such as $\boldsymbol{H}_{\text{CIRC}}$ can be given as [3],

$$\boldsymbol{H}_{\text{CIRC}} = \boldsymbol{F}^{\text{H}}\boldsymbol{\Lambda}\boldsymbol{F} \tag{6}$$

$\boldsymbol{F}$ being the unitary Fourier matrix and the diagonal matrix $\boldsymbol{\Lambda} \in \mathbb{C}^{N\times N}$ containing eigenvalues of the circulant matrix can be given in this case as

$$\boldsymbol{\Lambda} = \text{diag}\left(\boldsymbol{F}\begin{bmatrix}\boldsymbol{h}\\ \boldsymbol{0}_{N-L}\end{bmatrix}\right) \triangleq \boldsymbol{H} \tag{7}$$

The matrix $\boldsymbol{H} \in \mathbb{C}^{N\times N}$ is defined to be a diagonal matrix containing the *Channel Frequency Response (CFR)* coefficients along its main diagonal. Plugging in the substitutions from equations 6 and 7, the system model in equation 5 reduces to

$$\boldsymbol{Y}_k = \boldsymbol{H}\boldsymbol{X}_k + \eta_k \tag{8}$$

We note that the fading multipath channel boils down to a number of interference-free parallel sub-channels whereby, each of the received sub-carrier can be given as the corresponding transmitted sub-carrier scaled by a scalar complex fading coefficient (CFR at that sub-carrier) and corrupted by the additive noise. The detection scheme at the receiver can be as simple as just dividing the received sub-carrier by the estimated Channel Frequency Response. Schemes for estimation of channel, followed by data detection (in the context of DFT-SOFDM systems) can be found for instance in [4].

Presented below in Figure 2 is a comprehensive graphical representation of the OFDM system model, whereby we note the existence of interference-free parallel sub-channels in the frequency domain.

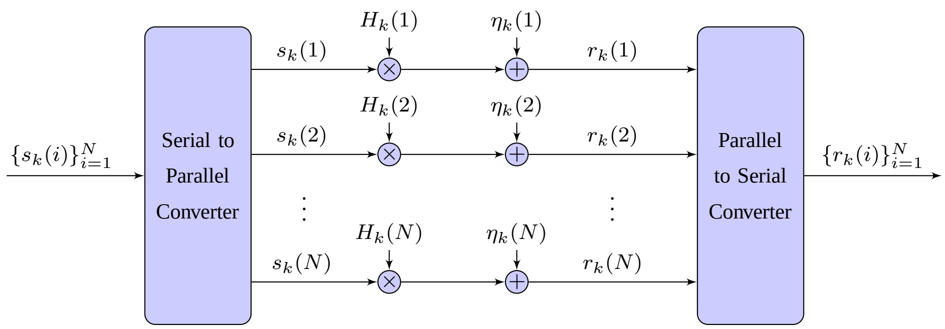


Fig. 2. Effective OFDM System Model with a cyclic prefix exceeding CIR length and the channel being constant during the transmission of one OFDM block.

## REFERENCES

[1] Andrea Goldsmith, *Wireless Communications*, Cambridge University Press, 2005.

[2] M. D. Nisar, W. Utscick, H. Nottensteiner, and T. Hindelang, "Channel Estimation and Equalization of OFDM Systems with Insufficient Cyclic Prefix," in *65th IEEE Vehicular Technology Conference*, Apr 2007.

[3] Robert M. Gray, "Toeplitz and Circulant Matrices: A review," *Deptartment of Electrical Engineering, Stanford University*, 1971.

[4] M. Danish Nisar, *Channel Estimation and Equalization for Evolved-UTRA Uplink*, MS Thesis, Munich University of Technology, TUM Germany, October 2006.